

How Do Glassy Domains Grow?

Saroj Kumar Nandi^{1,*} and Sriram Ramaswamy^{1,†}

¹*Centre for Condensed Matter Theory, Department of Physics,
Indian Institute of Science, Bangalore-560 012, India.*

We construct the equations for the growth kinetics of a structural glass within mode-coupling theory, through a non-stationary variant of the 3-density correlator defined in Phys. Rev. Lett. **97**, 195701 (2006). We solve a schematic form of the resulting equations to obtain the coarsening of the 3-point correlator $\chi_3(t, t_w)$ as a function of waiting time t_w . For a quench into the glass, we find that χ_3 attains a peak value $\sim t_w^{0.5}$ at $t - t_w \sim t_w^{0.8}$, providing a theoretical basis for the numerical observations of Parisi [J. Phys. Chem. B **103**, 4128 (1999)] and Kob and Barrat [Phys. Rev. Lett. **78**, 4581 (1997)]. The aging is not “simple”: the t_w dependence cannot be attributed to an evolving effective temperature.

When a system is quenched below an ordering transition, domains of the ordered phase appear and begin to grow [1], with characteristic size given by the decay length of equal-time correlations of the order parameter. The corresponding issue for the glass transition has been examined numerically [2–4], using susceptibilities and correlation lengths that capture the onset of amorphous freezing [2, 5–7], but a quantitative theory of these observations has been lacking [8]. Length-scale information similar to that obtained from the 4-density correlator and related overlap functions [2, 5, 6] has been shown [9] to be contained in a certain three-point correlator $\chi_3(t)$, whose peak value, and the time at which the peak is attained, diverge [9] upon approaching the mode-coupling glass transition [10].

In this work we present a theory of the *coarsening of glassy order*, using a non-stationary generalization $\chi_3(t, t_w)$, whose peak value $\Omega(t_w)$ is the correlation volume as a function of the waiting time t_w since the quench. We formulate our calculation in the framework of the fluctuating hydrodynamics of a dense liquid, and obtain results using mode-coupling theory (MCT) [10–12], in a schematic approach [13]. Figures 1 - 5 summarize our results. We find that $\Omega(t_w)$ grows without bound for a quench into the MCT glass (Fig. 2), as $t_w^{0.5}$, and the relaxation time as $t_w^{0.8}$ (Fig. 5), in agreement respectively with the computer experiments of Parisi [2] and Kob and Barrat [4]. As effects beyond MCT cut off the transition, the coarsening in experiments, simulations, or a complete theory will cease at long enough times, but typical simulations do not explore these asymptotically long time scales and can therefore be compared usefully to our MCT coarsening predictions. The three-point function, Fig. 5, shows features incompatible with “simple aging” [4, 14] but qualitatively similar to [3]. For a quench to a distance ϵ from the threshold value on the *liquid* side, Ω grows to saturation (Fig. 3), reaching an equilibrium value $\sim \epsilon^{-1}$, with a relaxation time $\sim \epsilon^{-1.8}$ (Figure 4).

MCT is an analytically tractable approximation to equilibrium liquid-state dynamics that yields a glass transition in a homogeneous system, which is why it is so widely used despite its shortcomings [7, 11]. In or-

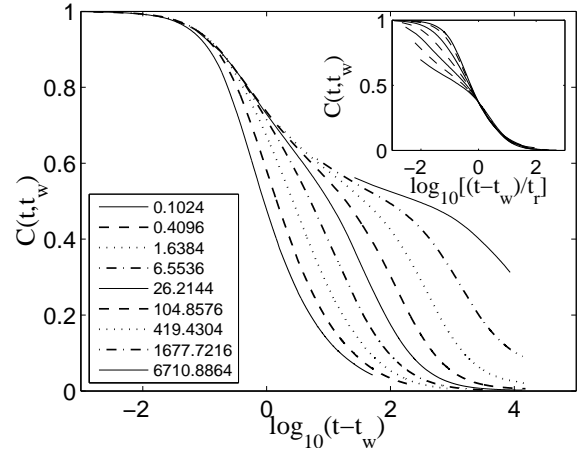


FIG. 1: The aging of the two-point function. The correlation function $C(t, t_w)$ as a function of t , for various waiting times t_w shown in the legend. The decay with t becomes progressively slower with increasing t_w . The final parameter values are $T = 1.0$ and $\lambda = 2.0$. **Inset:** Scaling $t - t_w$ by t_r yields a data collapse in the α -relaxation regime. Such “simple aging”, however, is not seen in the three-point function, Fig. 5.

der to extend MCT to describe non-stationary states such as coarsening we work with a general field-theoretic approach [11, 15], taking care not to use results like the Kubo formulae and fluctuation-dissipation relations (FDR), which are justified only in equilibrium treatments [16, 17]. We start with the equations of fluctuating hydrodynamics for the velocity and density fields for an isothermal compressible fluid, extended to large wavenumbers so as to take into account the modes around the structure factor peak [10]. In order to obtain an equation for the density field alone, we eliminate the velocity while retaining momentum conservation but ignoring inertia. This yields the dynamical equation

$$\frac{\partial \delta \rho_k(t)}{\partial t} + K_1 \delta \rho_k(t) = \frac{K_2}{2} \int_{\mathbf{q}} \mathcal{V}_{k,q} \delta \rho_q(t) \delta \rho_{k-q}(t) + f_k(t), \quad (1)$$

for the Fourier-transformed density fluctuation $\delta \rho_k(t)$

at wavevector \mathbf{k} , with $\mathcal{V}_{k,q} = \mathbf{k} \cdot [\mathbf{q}c_q + (\mathbf{k} - \mathbf{q})c_{k-q}]$, $K_1 = k_B T / S_k D_L$ and $K_2 = k_B T / D_L k^2$. Eq. (1) can be viewed as the no-inertia limit of Eq. (4.1) of [16]. Here $D_L = (\zeta + 4\eta/3)/\rho_0$ is the longitudinal damping, where ζ and η are the bare shear and bulk viscosities, $k_B T$ is Boltzmann's constant times temperature, S_k and c_k are the equilibrium static structure factor and direct correlation function respectively and the noise $f_k(t)$ obeys

$$\langle f_k(t)f_{k'}(t') \rangle = \frac{2k_B T}{D_L} \rho_k(t) \delta(k + k') \delta(t - t'). \quad (2)$$

From diagrammatic perturbation theory [11, 15], we construct the equations of motion for the correlation function $C_k(t, t_w) = \langle \delta\rho_k(t) \delta\rho_{-k}(t_w) \rangle$ and response function $R_k(t, t_w) = \langle \partial\delta\rho_k(t) / \partial f_{-k}(t_w) \rangle$:

$$\begin{aligned} \frac{\partial C_k(t, t_w)}{\partial t} = & -K_1 C_k(t, t_w) + \int_0^{t_w} ds D_k(t, s) R_k(t_w, s) \\ & + \int_0^t ds \Sigma_k(t, s) C_k(s, t_w), \end{aligned} \quad (3a)$$

$$\begin{aligned} \frac{\partial R_k(t, t_w)}{\partial t} = & \delta(t - t_w) - K_1 R_k(t, t_w) \\ & + \int_{t_w}^t ds \Sigma_k(t, s) R_k(s, t_w), \end{aligned} \quad (3b)$$

with $D_k(t, t') = (2k_B T / D_L) \rho_k(t) \delta(t - t') + M_k(t, t')$, $M_k(t, t') = (K_2^2 / 2) \int_{\mathbf{q}} \mathcal{V}_{k,q}^2 C_q(t, t') C_{k-q}(t, t')$ and $\Sigma_k(t, t') = K_2^2 \int_{\mathbf{q}} \mathcal{V}_{k,q}^2 R_q(t, t') C_{k-q}(t, t')$. The contribution to (3a) from the first term in $D_k(t, t')$ vanishes due to causality. Franz and Hertz [13] obtained schematic equations similar to (3a) and (3b) for the Amit-Roginsky model [18].

How are the input quantities K_1 and $\mathcal{V}_{k,q}$ in equations (3a) and (3b) defined for the case of a quench? A comparison with the treatment of Zaccarelli *et al.* [17] is useful here. The $\mathcal{V}_{k,q}$ term in (1) and (3) involves the “residual interactions” in [17]. We define our quench to be an abrupt increase in the interaction strength, implying that $\mathcal{V}_{k,q}$ should be evaluated at the final parameter value. To determine K_1 , which must now be a time-dependent quantity as we are dealing with a non-stationary state, we insist, as in [19], that for $\tau = (t - t_w) \ll t_w$ Eq. (3) obeys time-translation invariance and the FDR. This leads, after some algebra, to

$$\begin{aligned} K_1(t) S_k = & T R_k(0) + K_2^2 \int_0^t \int_{\mathbf{q}} \mathcal{V}_{k,q}^2 C_{k-q}(t, s) \times \\ & \left[\frac{1}{2} C_q(t, s) R_k(t, s) + R_q(t, s) C_k(t, s) \right] ds. \end{aligned} \quad (4)$$

In [17] the term corresponding to K_1 enters through the equal-time density correlator. The latter being time-dependent in a coarsening situation, it is natural that K_1 should change in time.

To obtain the equation of motion for the growth kinetics of glassy correlations, we look at the behaviour of

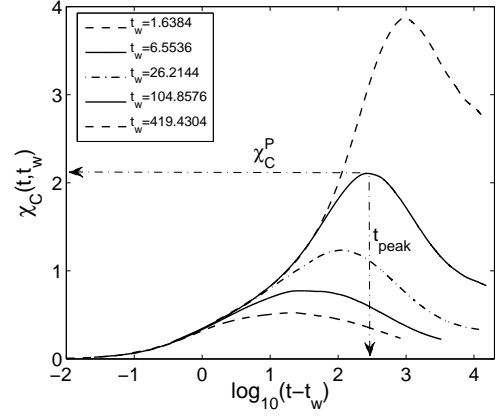


FIG. 2: The growth kinetics of glassy correlations. The three-point correlation function in an aging structural glassy system, from schematic mode-coupling theory. The peak value of $\chi_C(t, t_w)$ grows and shifts to higher t_{peak} with increasing waiting time t_w .

our nonstationary generalization of the three-density correlation function [9] mentioned above. We introduce in the free-energy functional a one-body term $\epsilon(\mathbf{r})\rho(\mathbf{r})$, coupling the density to an external potential $\epsilon(\mathbf{r})$ and leading on average to an inhomogeneous shift $\delta m(\mathbf{r})$ in the mean density field [20]. We work in the limit where $\epsilon(\mathbf{r})$ and hence $\delta m(\mathbf{r})$ are uniform, so that the Fourier transform $\delta m_{\mathbf{k}}$ has non-zero weight δm_0 only for wavevector $\mathbf{k} = 0$. We will see that this suffices for the purpose of extracting the correlation volume. The resulting generalized Langevin equation for ρ , to first order in the background density δm_0 which encodes the effects of the field, is [21, 22]

$$\begin{aligned} \frac{\partial \delta\rho_k(t)}{\partial t} + K_1(t) \delta\rho_k(t) - \frac{k_B T c_k \delta m_0}{D_L} \delta\rho_k(t) \\ = \frac{K_2}{2} \int_{\mathbf{q}} \mathcal{V}_{k,q} \delta\rho_q(t) \delta\rho_{k-q}(t) + f_k(t). \end{aligned} \quad (5)$$

Let $\tilde{C}_k(t, t_w)$ and $\tilde{R}_k(t, t_w)$ denote the δm_0 -dependent two-point correlation and response functions implied by (5). As we are working in a non-stationary state, we must define separate 3-point quantities analogous to χ_3 in [9] for \tilde{C}_k and \tilde{R}_k : $\chi_k^C(t, t_w) = \partial \tilde{C}_k(t, t_w) / \partial \delta m_0|_{\delta m_0 \rightarrow 0}$ and $\chi_k^R(t, t_w) = \partial \tilde{R}_k(t, t_w) / \partial \delta m_0|_{\delta m_0 \rightarrow 0}$, with equations of

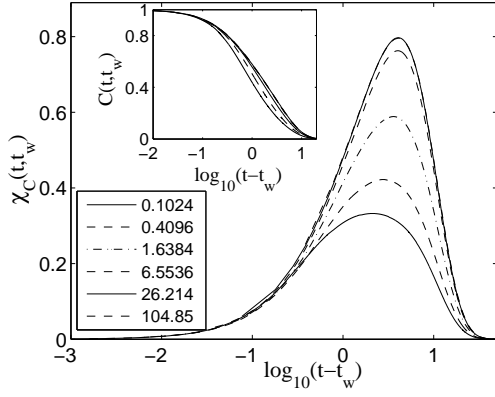


FIG. 3: Evolution of three-point function following a quench to a point close to but on the liquid side of the transition. Growth to saturation of $\chi_C(t, t_w)$ as a function of t for various t_w for $\lambda = 0.75$, corresponding to a quench into the liquid. **Inset:** The two-point correlator, for the same parameter values, shows a relaxation time that grows progressively with increasing t_w but approaches a finite value.

motion

$$\frac{\partial \chi_k^R(t, t_w)}{\partial t} + K_1(t) \chi_k^R(t, t_w) = \int_{t_w}^t ds \Sigma_k(t, s) \chi_k^R(s, t_w) + \int_{t_w}^t ds \tilde{\Sigma}'_k(t, s) R_k(s, t_w) + \mathcal{S}_k^R(t, t_w), \quad (6a)$$

$$\frac{\partial \chi_k^C(t, t_w)}{\partial t} + K_1(t) \chi_k^C(t, t_w) = \int_0^{t_w} ds M_k(t, s) \chi_k^R(t_w, s) + \int_0^{t_w} ds \tilde{M}'_k(t, s) R_k(t_w, s) + \int_0^t ds \Sigma_k(t, s) \chi_k^C(s, t_w) + \int_0^t ds \tilde{\Sigma}'_k(t, s) C_k(s, t_w) + \mathcal{S}_k^C(t, t_w), \quad (6b)$$

where $\tilde{\Sigma}'_k(t, s) = \partial \tilde{\Sigma}_k(t, s) / \partial \delta m_0|_{\delta m_0 \rightarrow 0}$, and $\tilde{M}'_k(t, s) = \partial \tilde{M}_k(t, s) / \partial \delta m_0|_{\delta m_0 \rightarrow 0}$. $\tilde{M}_k(t, s)$ and $\tilde{\Sigma}_k(t, s)$ are quantities corresponding to M and Σ but evaluated in the presence of $\epsilon(\mathbf{r})$. The expressions for the source terms $\mathcal{S}_k^R(t, t_w)$ and $\mathcal{S}_k^C(t, t_w)$ are given in the supplementary information (SI) and the straightforward but tedious derivation of (6a) and (6b) will be presented in a subsequent paper [20].

Simplified integral equations keeping track of time-dependence alone [23–25] have proved invaluable in extracting meaningful results from MCT within a manageable calculation. In this spirit, we suppress dependence on wavevector \mathbf{k} and write the self-energies in (3) and (6) as $M(t, s) = 2\lambda C^2(t, s)$ and $\Sigma(t, s) = 4\lambda R(t, s)C(t, s)$, yielding equations for the two- and three-point correlation and response functions which we now denote $C(t, t_w)$, $R(t, t_w)$, $\chi_C(t, t_w)$ and $\chi_R(t, t_w)$. We solve the resulting schematic versions of (3), (4) and (6), whose detailed forms are given in the SI section, using the algorithm developed by Kim and Latz [26, 27], to give the results quoted at the start of this paper. First, aging

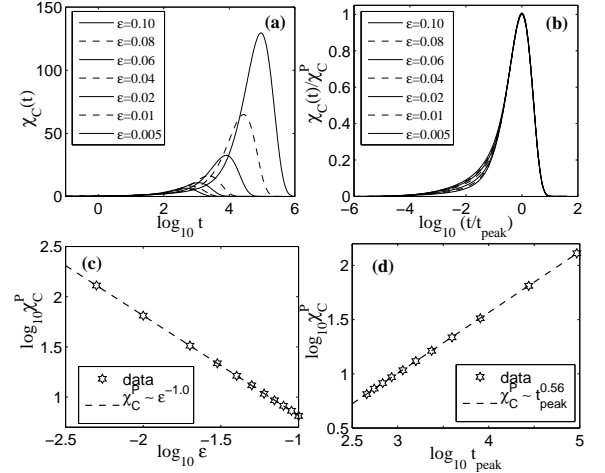


FIG. 4: Three-point function and correlation volume in equilibrium. Three-point function and correlation volume in equilibrium. (a) $\chi_C(t)$ as a function of t for various $\epsilon \equiv |\lambda - \lambda_c|$. (b) When $\chi_C(t)$ is scaled with χ_C^P and time with t_{peak} , we obtain data collapse for large t . (c) $\chi_C^P \sim \epsilon^{-1.0}$. (d) $\chi_C^P \sim t_{peak}^{0.56}$

can clearly be seen in the behaviour of $C(t, t_w)$ in Fig. 1. Second and more important is the characteristic non-monotone behavior of $\chi_C(t, t_w)$, and its dependence on t_w and interaction strength λ (Fig. 2). For a fixed initial condition corresponding in our schematic approach to a liquid with negligible correlations, we examine in particular how $\chi_C(t, t_w)$ as a function of t changes with t_w , for values of λ corresponding to the liquid and the glass phase. Recall that λ defines the point to which the system is quenched. For λ in the liquid phase but close to the transition we find, as expected, that $\chi_C(t, t_w)$ attains a peak value χ_C^P at a time t_{peak} , with both χ_C^P and t_{peak} growing with t_w but saturating to finite values as shown in Fig. 3. The final peak value of $\chi_C(t, t_w)$ grows as $(\lambda - \lambda_c)^{-1}$ and $\chi_C^P \sim t_{peak}^{0.56}$ (Fig. 4). These final values, obtained at $t_w \rightarrow \infty$ are the equilibrium values of the corresponding quantities. In the notation of [9] we are working at $\mathbf{q}_0 \rightarrow 0$ and our results are consistent with theirs in that limit. A more detailed comparison with [9] or [28], including an estimate of the correlation *length* requires a calculation of the sensitivity of two-point functions to a spatially varying potential.

For a quench into the glassy region, $\lambda = 2.0$, as shown in Fig. 2, χ_C^P grows without bound. In more detail (Fig. 5), $\chi_C^P \sim t_w^a$ at a time $t_{peak} \sim t_w^b$, with increasing t_w . The exponents $a \simeq 0.5$ in agreement with simulations [2] and $b \simeq 0.8$. From the two-point correlator $C(t, t_w)$ we find a relaxation time $t_r \simeq t_{peak}/4$ close to the transition. Thus our result implies $t_r \sim t_w^{0.8}$, in agreement with the numerical experiment of Ref. [4]; the relation between t_r and t_{peak} remains to be tested. χ_C^P measures an effective correlation volume, so that its growth is the claimed coarsening of glassy structure, and is consistent

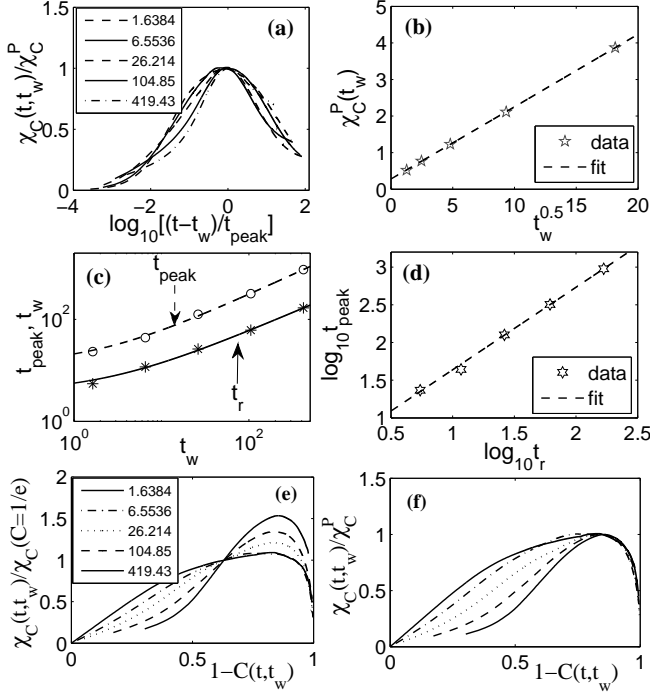


FIG. 5: Scaling of order-parameter correlations and relaxation time following a quench into the glass: no simple aging. (a) If we scale $\chi_C(t, t_w)$ by $\chi_C^P(t_w)$ and time by t_{peak} , the three-point function does not follow a master curve: the behaviour at various t_w is not the equilibrium dynamics at an evolving effective temperature. This behaviour of χ_C belies the expectation of “simple aging” suggested by the two-point function in Fig. 1. (b) The peak height $\chi_C^P(t_w) \propto t_w^{1/2}$. (c) t_{peak} and t_r both grow as $t_w^{0.8}$. (d) t_{peak} is proportional to t_r . (e) If we scale $\chi_C(t, t_w)$ by $\chi_C(C = 1/e)$ and plot it as a function of $1 - C(t, t_w)$, as in Parsaeian *et al.* [3], no data collapse is found. t_w values are shown in the legend. (f) Scaling $\chi_C(t, t_w)$ by $\chi_C^P(t_w)$ and plotting it as a function of $1 - C(t, t_w)$ shows data collapse in the regime of α -relaxation. Waiting times as in (e).

with the idea of a growing “domain size”. Regardless of the precise values obtained, it is significant that our theory and the simulations of [2–4] all find a total structure factor χ_C^P growing very sublinearly in time. Our scaling laws differ quantitatively (Fig. 5e) from those of [3], perhaps because we measure different quantities. However, if we scale $\chi_C(t, t_w)$ by $\chi_C^P(t_w)$ and plot them as a function of $1 - C(t, t_w)$, data collapse is obtained in the α -relaxation regime (Fig. 5f). We do not claim to understand the origin of this scaling or, for that matter, that of Ref. [3]. A similar calculation [20] for the three-point correlation function for a p -spin spin-glass model with $p = 3$ finds again a growing χ_C^P , but slower than for the present problem.

We emphasize that the t_w -dependent properties we extract do not correspond to those of an equilibrium system at an evolving λ or temperature. Had it been so, scaling $\chi_C(t, t_w)$ by $\chi_C^P(t_w)$ and time by t_{peak} would have given

data collapse for all t_w as for the equilibrium case (Fig. 4). Fig. 5 shows the absence of such collapse even for larger t_w . It would appear that the 3-point correlator is more sensitive to departures from “simple aging”, and an interpretation in terms of an evolving effective temperature, than the two-time correlation function [4, 14] (see inset of Fig. 1). Perhaps the monotone decay of the latter masks such deviations or, more likely, χ_C carries additional, independent information.

We close by summarising the achievements of this work. We have shown that mode-coupling theory adapted to describe non-stationary states captures the key features of the emergence and coarsening of glassy order from a liquid. Through the evolution of a three-point function we have shown that the glassy correlation volume grows as $t_w^{0.5}$ with waiting time t_w , slower than domain volumes in conventional coarsening, and the relaxation time of the glass grows as $t_w^{0.8}$. These theoretical growth laws are supported by simulation studies [2, 4], and the broad features we observe are similar to those in [3]. In an experimental realization, if the quench is below the MCT transition but above a putative ideal glass transition at, say, the Kauzmann temperature T_K , activated processes [29] outside the scope of MCT should cut off the growth. Presumably a quench below T_K will give indefinite growth of a different glassy length scale [30–32] with a form not predicted by MCT. In results to be presented separately [20] we find further that an imposed shear-rate $\dot{\gamma}$ cuts off aging and coarsening at $t_w \sim 1/\dot{\gamma}$ in the glassy region and $t_w = \min(t_r, 1/\dot{\gamma})$ in the fluid. Since the relaxation time goes as $t_w^{0.8}$, this should imply that t_r or t_{peak} should vary as $\dot{\gamma}^{-0.8}$. We look forward to experimental tests of our results.

We thank C. Dasgupta for valuable suggestions and a critical reading of our paper and N. Menon for enlightening comments. We also thank S.M. Bhattacharyya, B. Kim, K. Miyazaki, S. Sastry, D. Sen, S.P. Singh and E. Zaccarelli for discussions. SKN was supported in part by the University Grants Commission and SR by a J.C. Bose Fellowship from the Department of Science and Technology.

Supplementary information (SI)

The schematic form of the equations and the source terms

The schematic form of the equations (3a) and (3b), is obtained by following the outline given in the paper. The

final forms will be:

$$\begin{aligned}\frac{\partial C(t, t_w)}{\partial t} &= -\mu(t)C(t, t_w) + 2\lambda \int_0^{t_w} C^2(t, s)R(t_w, s)ds \\ &\quad + 4\lambda \int_0^t C(t, s)R(t, s)C(s, t_w)ds, \\ \frac{\partial R(t, t_w)}{\partial t} &= \delta(t - t_w) - \mu(t)R(t, t_w) \\ &\quad + 4\lambda \int_{t_w}^t R(t, s)C(t, s)R(s, t_w)ds\end{aligned}\quad (7)$$

where $\mu(t)$ is the schematic version of $K_1(t)$:

$$\mu(t) = T + 6\lambda \int_0^t C^2(t, s)R(t, s)ds. \quad (8)$$

Eqs. (7a) and (7b) contain two “source” terms which we present in detail here. To proceed, let us define

$$\begin{aligned}\omega_k(t) &= \frac{K_2^2}{S_k} \int_0^t \int_{\mathbf{q}} \mathcal{V}_{k,q}^2 \left[\chi_{k-q}^C(t, s) \left\{ \frac{1}{2} C_q(t, s) R_k(t, s) \right. \right. \\ &\quad \left. \left. + R_q(t, s) C_k(t, s) \right\} ds \right. \\ &\quad \left. + C_{k-q}(t, s) \left\{ \frac{1}{2} \chi_q^C(t, s) R_k(t, s) + \frac{1}{2} C_q(t, s) \chi_k^R(t, s) \right. \right. \\ &\quad \left. \left. + \chi_q^R(t, s) C_k(t, s) + R_q(t, s) \chi_k^C(t, s) \right\} \right] ds. \quad (9)\end{aligned}$$

Then the source terms can be written in the form

$$\begin{aligned}\mathcal{S}_k^R(t, t_w) &= \frac{k_B T c_k}{D_L} R_k(t, t_w) - \omega_k(t) R_k(t, t_w), \\ \mathcal{S}_k^C(t, t_w) &= \frac{k_B T c_k}{D_L} C_k(t, t_w) - \omega_k(t) C_k(t, t_w)\end{aligned}\quad (10)$$

The equations for the three-point correlators are also schematicised in a similar way as stated in the paper. The final schematic forms of equations (7a) and (7b) will be

$$\begin{aligned}\frac{\partial \chi_R(t, t_w)}{\partial t} + \mu(t) \chi_R(t, t_w) &= 4\lambda \int_{t_w}^t R(t, s) C(t, s) \chi_R(s, t_w) ds + 4\lambda \int_{t_w}^t R(t, s) \chi_C(t, s) R(s, t_w) ds \\ &\quad + 4\lambda \int_{t_w}^t \chi_R(t, s) C(t, s) R(s, t_w) ds + \mathcal{S}_R(t, t_w)\end{aligned}\quad (11)$$

$$\begin{aligned}\frac{\partial \chi_C(t, t_w)}{\partial t} + \mu(t) \chi_C(t, t_w) &= 4\lambda \int_0^{t_w} C(t, s) \chi_C(t, s) R(t_w, s) ds + 2\lambda \int_0^{t_w} C^2(t, s) \chi_R(t_w, s) ds \\ &\quad + 4\lambda \int_0^t C(t, s) R(t, s) \chi_C(s, t_w) ds + 4\lambda \int_0^t \chi_C(t, s) R(t, s) C(s, t_w) ds \\ &\quad + 4\lambda \int_0^t C(t, s) \chi_R(t, s) C(s, t_w) ds + \mathcal{S}_C(t, t_w)\end{aligned}\quad (12)$$

with the source terms given as $\mathcal{S}_R(t, t_w) = [1 - \omega(t)]R(t, t_w)$ and $\mathcal{S}_C(t, t_w) = [1 - \omega(t)]C(t, t_w)$ where $\omega(t)$, the schematic form of $\omega_k(t)$, is given as

$$\begin{aligned}\omega(t) &= 12\lambda \int_0^t C(t, s) \chi_C(t, s) R(t, s) ds \\ &\quad + 6\lambda \int_0^t C^2(t, s) \chi_R(t, s) ds.\end{aligned}\quad (13)$$

* Electronic address: snandi@physics.iisc.ernet.in

† Electronic address: sriram@tifrh.res.in; On leave at TIFR Centre for Interdisciplinary Sciences, 21 Brundavan Colony, Narsingi, Hyderabad 500 075, India.

[1] A. J. Bray, Adv. Phys. **43**, 357 (1994).

[2] G. Parisi, J. Phys. Chem. B **103**, 4128 (1999).

[3] A. Parsaeian and H. E. Castillo, Phys. Rev. E **78**, 060105

- (2008).
- [4] W. Kob and J.-L. Barrat, Phys. Rev. Lett. **78**, 4581 (1997).
 - [5] C. Dasgupta, A. V. Indrani, S. Ramaswamy, and M. K. Phani, Europhys. Lett. **15**, 307 (1991).
 - [6] L. Berthier, G. Biroli, J.-P. Bouchaud, L. Cipelletti, D. E. Masri, D. L'Hôte, F. Ladieu, and M. Pierno, Science **310**, 1797 (2005).
 - [7] L. Berthier and G. Biroli, Rev. Mod. Phys. **83**, 587 (2011).
 - [8] The aging of the time-decay of two-point density correlations has been studied in experiments [33], numerical simulations [4], mode-coupling theories [34] and generalized Langevin treatments [35] on structural glasses, and in related theoretical work on spin-glasses [14, 19, 26].
 - [9] G. Biroli, J.-P. Bouchaud, K. Miyazaki, and D. R. Reichman, Phys. Rev. Lett. **97**, 195701 (2006).
 - [10] S. P. Das, Rev. Mod. Phys. **76**, 785 (2004).
 - [11] D. Reichman and P. Charbonneau, J. Stat. Mech. pp. P05013–1–P05013–23 (2005).
 - [12] W. Götze, *Complex Dynamics of Glass-Forming Liquids: A mode-coupling theory* (Oxford University Press, 2009).
 - [13] S. Franz and J. Hertz, Phys. Rev. Lett. **74**, 2114 (1995).
 - [14] S. P. Singh and S. P. Das, Phys. Rev. E **79**, 031504 (2009).
 - [15] J. P. Bouchaud, L. Cugliandolo, J. Kurchan, and M. Mézard, Physica A **226**, 243 (1996).
 - [16] K. Kawasaki, J. Stat. Phys. **110**, 1249 (2003).
 - [17] E. Zaccarelli, G. Foffi, P. D. Gregorio, F. Sciortino, P. Tartaglia, and K. A. Dawson, J. Phys.: Condens. Matter **14**, 2413 (2002).
 - [18] D. Amit and D. Roginsky, J. Phys. A: Math. Gen. **12**, 689 (1979).
 - [19] L. F. Cugliandolo and J. Kurchan, Phys. Rev. Lett. **71**, 173 (1993).
 - [20] S. K. Nandi and S. Ramaswamy, in preparation (2012).
 - [21] S. K. Nandi, S. M. Bhattacharyya, and S. Ramaswamy, Phys. Rev. E **84**, 061501 (2011).
 - [22] We introduce the external potential in the coarse-grained free-energy functional whereas [9] do so in the microscopic Hamiltonian, where it must affect all degrees of freedom, fast and slow.
 - [23] J. Brader, T. Voigtmann, M. Fuchs, R. Larson, and M. Cates, Proc. Natl. Acad. Sci. USA **106**, 15186 (2009).
 - [24] T. Kirkpatrick, Phys. Rev. A **31**, 939 (1985).
 - [25] E. Leutheusser, Phys. Rev. A **29**, 2765 (1984).
 - [26] B. Kim and A. Latz, Europhys. Lett. **53**, 660 (2001).
 - [27] D. Herzbach, Master's thesis, Institut für Physik, Johannes Gutenberg Universität, Mainz (2000).
 - [28] S. Karmakar, C. Dasgupta, and S. Sastry, PNAS **106**, 3675 (2009).
 - [29] S. M. Bhattacharyya, B. Bagchi, and P. G. Wolynes, Proc. Natl. Acad. Sci. USA **105**, 16077 (2008).
 - [30] J. Kurchan and D. Levine, J. Phys. A: Math. Theor. **44**, 035001 (2011).
 - [31] C. Cammarota and G. Biroli, ArXiv e-prints (2011), 1106.5513.
 - [32] L. Berthier and W. Kob, Phys. Rev. E **85**, 011102 (2012).
 - [33] D. Bonn, S. Tanase, B. Abou, H. Tanaka, and J. Meunier, Phys. Rev. Lett. **89**, 015701 (2002).
 - [34] A. Latz, J. Phys.: Condens. Matter **12**, 6353 (2000).
 - [35] P. Ramírez-González and M. Medina-Noyola, Phys. Rev. E **82**, 061504 (2010).